NASA TT F-10,786

ON THE PROBLEM OF ELASTIC EQUILIBRIUM OF AN ELLIPSOID OF REVOLUTION

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Translation of "Sul problema dell'equilibrio elastico di un ellissoide di rotazione".
Atti della Reale Accademia dei Lincei. Classe di scienze fisiche, matematiche e naturali, Rendiconti, Ser. 5, Vol. 14, pp. 76-84, 1905.

NASA TTF-10,7 86

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON D.C. MARCH 1967

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The subject problem is analyzed for the case where the surface displacements are given using the theory of Green's functions. The case where the surface stresses are specified is briefly discussed.

1. *Case in which the displacements on the surface are given. Introductory formulas.* Let us remember that, in accord with the principles which we have applied several times, when surface displacements in an elastic isotropic deformed body in equilibrium are known the displacements u, v, w in an internal point in the body itself are given by the formulas: /76**

$$\left\{\begin{aligned} u &= \frac{1}{4\pi}\int_\sigma u\frac{dG}{dn}\,d\sigma - \frac{\lambda+\mu}{2\mu}x\theta + \frac{\lambda+\mu}{8\pi\mu}\int_\sigma \xi\theta\frac{dG}{dn}\,d\sigma, \\ v &= \frac{1}{4\pi}\int_\sigma v\frac{dG}{dn}\,d\sigma - \frac{\lambda+\mu}{2\mu}y\theta + \frac{\lambda+\mu}{8\pi\mu}\int_\sigma \eta\theta\frac{dG}{dn}\,d\sigma, \\ w &= \frac{1}{4\pi}\int_\sigma w\frac{dG}{dn}\,d\sigma - \frac{\lambda+\mu}{2\mu}z\theta + \frac{\lambda+\mu}{8\pi\mu}\int_\sigma \zeta\theta\frac{dG}{dn}\,d\sigma, \end{aligned}\right. \tag{1}$$

it is assumed that the value of the harmonic function θ is known which is inherent in the problem. As is known, it represents the elementary expansion; here σ is the surface of the elastic body; n, the internal normal; G, the ordinary Green function; λ and μ are the usual Lamé constants; and ξ, η, ζ are the values of x, y, z on σ. In solving the problem, which we shall try to do rapidly, we will make no use of the effective value of the Green function, but we will only use this to give a convenient representation to a harmonic function which takes on given values on the surface. If equations (1) are constructed in any manner on the assumption that θ is known, for the complete solution of the problem, we must determine θ from equation

$$\left\{\begin{aligned} &\frac{3\lambda+5\mu}{2\mu}\theta + \frac{\lambda+\mu}{2\mu}\left(x\frac{\partial\theta}{\partial x} + y\frac{\partial\theta}{\partial y} + z\frac{\partial\theta}{\partial z}\right) - \\ &-\frac{\lambda+\mu}{8\pi\mu}\left(\frac{\partial}{\partial x}\int_\sigma \xi\theta\frac{dG}{dn}\,d\sigma + \frac{\partial}{\partial y}\int_\sigma \eta\theta\frac{dG}{dn}\,d\sigma + \frac{\partial}{\partial z}\int_\sigma \zeta\theta\frac{dG}{dn}\,d\sigma\right) = \\ &= \frac{1}{4\pi}\left(\frac{\partial}{\partial x}\int_\sigma u\frac{dG}{dn}\,d\sigma + \frac{\partial}{\partial y}\int_\sigma v\frac{dG}{dn}\,d\sigma + \frac{\partial}{\partial z}\int_\sigma w\frac{dG}{dn}\,d\sigma\right). \end{aligned}\right. \tag{2}$$

Arriving at our special case, we formulate the equation of the ellipsoid in the form

$$\frac{x^2}{r^2} + \frac{y^2+z^2}{r^2-1} = h^2 \tag{3}$$

where r and h are assumed to be real if the ellipsoid is elongated with $|r| > 1$. If the ellipsoid is compressed, r and h are assumed to be purely imaginary. /77

*Presented by V. Tolterra, Society member.

**Numbers in the margin indicate pagination in the original foreign text.

When in any case the following are then written:

$$x = h\rho t \ , \ y = h\sqrt{(\rho^2-1)(1-t^2)}\cos\psi \ , \ z = h\sqrt{(\rho^2-1)(1-t^2)}\,\text{sen}^{*}\psi \tag{4}$$

ρ, t, ψ will be a system of orthogonal curvilinear coordinates.(1) In the case of an elongated ellipsoid, we will--in addition to assuming h to be real--also hypothesize that ρ is real with $|\rho| \geq 1$ and that $|t| \leq 1$. In the case of a compressed ellipsoid, in addition to assuming that h is real, we will hypothesize likewise that ρ is purely imaginary and that $|t| \leq 1$. With these conventions, in the coordinate system ρ, t, ψ, equation (3) of the ellipsoid will always be $\rho = r$.

If we then set the values of any one function ϕ of the points of surface (3) satisfying the known general conditions--which we will not pause to present here--in the form

$$\sum_0^\infty{}_i \sum_i^\infty{}_m (A_{m,i}\cos i\psi + B_{m,i}\,\text{sen}\, i\psi)\, P_{m,i}(t) \tag{5}$$

where

$$P_{m,i}(t) = (1-t^2)^{\frac{i}{2}} \frac{d^i P_m(t)}{dt^i} \quad ; \quad P_{m,0}(t) = P_m(t)$$

and the values of $P_m(t)$ are the ordinary Legendre functions, then, as is known, the following will be true:

$$\begin{cases} \dfrac{4\pi}{2m+1} A_{m,0} = \displaystyle\int_{-1}^{+1}\int_0^{2\pi} d\tau\, d\omega\, \varphi\, P_m(t), \\ \dfrac{4\pi}{2m+1} A_{m,i} = 2\dfrac{(m-i)!}{(m+i)!}\displaystyle\int_{-1}^{+1}\int_0^{2\pi} d\tau\, d\omega\, \varphi\, P_{m,i}(\tau)\cos i\omega \\ \dfrac{4\pi}{2m+1} B_{m,i} = 2\dfrac{(m-i)!}{(m+i)!}\displaystyle\int_{-1}^{+1}\int_0^{2\pi} d\tau\, d\omega\, \varphi\, P_{m,i}(\tau)\,\text{sen}\, i\omega, \end{cases} \tag{6}$$

The function Φ, which is harmonic and regular inside the ellipsoid which acquires the values of ϕ, will be given by the formula

$$\Phi = \sum_0^\infty{}_i \sum_i^\infty{}_m (A_{m,i}\cos i\psi + B_{m,i}\,\text{sen}\, i\psi) \frac{P_{m,i}(\rho)}{P_{m,i}(r)} P_{m,i}(t). \tag{7}$$

In both the cases in question--i.e., the case of the elongated ellipsoid and the case of the compressed ellipsoid--we conventionally write the following:

$$\sqrt{1-\rho^2} = \sqrt{-1}\sqrt{\rho^2-1} \quad , \quad \sqrt{1-r^2} = \sqrt{-1}\sqrt{r^2-1}.$$

It follows from this that in both these cases $P_{m,i}(\rho)$ and $P_{m,i}(r)$ are well determined functions such that their quotient is real. Moreover, $P_{m,i}(r)$ differs from zero for any value of m and of i.

To formulate the solution to our problem in the easiest way, let us note the following forms which we will use almost exclusively and which are, at least in part, well-known: /78

* Translator's note: Sen is properly sin in English terminology.

(1) If we decide to give only positive values to ρ if it is real, or positive values to the coefficient of the imaginary part if it is imaginary--while t may take on positive and negative values and ψ varies between zero and 2π--a single point in space will correspond to any system of values of ρ,t,ψ, and vice versa.

$$\begin{cases}(1-t^2)\,\dfrac{dP_m(t)}{dt} = (m+1)\,[tP_m(t) - P_{m+1}(t)], \\ (2m+1)\,tP_m(t) = (m+1)\,P_{m+1}(t) + mP_{m-1}(t), \qquad tP_0(t) = P_1(t) \\ (2m+1)\,P_m(t) = \dfrac{d}{dt}\,[P_{m+1}(t) - P_{m-1}(t)], \qquad P_0(t) = \dfrac{dP_1(t)}{dt}. \end{cases} \tag{8}$$

from which we easily deduce the others:

$$\begin{cases}(1-t^2)\,\dfrac{dP_{m,i}(t)}{dt} = -\,mtP_m(t) + (m+i)\,P_{m-1,i}(t), \\ (2m+1)\,tP_{m,i}(t) = (m-i+1)\,P_{m+1,i}(t) + (m+i)\,P_{m-1,i}(t), \\ (2m+1)\sqrt{1-t^2}\,P_{m,i}(t) = P_{m+1,i+1}(t) - P_{m-1,i+1}(t) = \\ = -(m-i+1)\,(m-i+2)\,P_{m+1,i-1}(t) + (m+i)\,(m+i-1)\,P_{m-1,i-1}(t), \\ t\sqrt{1-t^2}\,\dfrac{dP_{m,i}(t)}{dt} = m\sqrt{1-t^2}\,P_{m,i}(t) - i\,\dfrac{P_{m,i}(t)}{\sqrt{1-t^2}} + P_{m-1,i+1}(t), \\ t\sqrt{1-t^2}\,\dfrac{dP_{m,i}(t)}{dt} = m\sqrt{1-t^2}\,P_{m,i}(t) + i\,\dfrac{P_{m,i}(t)}{\sqrt{1-t^2}} - \\ \qquad -(m+i)\,(m+i-1)\,P_{m-1,i-1}(t), \\ \dfrac{P_{m+2,i}(\varrho)\,P_{m,i}(t) - P_{m+2,i}(t)\,P_{m,i}(\varrho)}{\varrho^2 - t^2} = (2m+3)\displaystyle\sum_{0}^{\left[\frac{m-i}{2}\right]}{}_k\,(2m+1-4k)\,\times \\ \qquad \times\,\dfrac{(m-i-2k)!}{(m-i+2)!}\,\dfrac{(m+i)!}{(m+i-2k)!}\,P_{m-2k,i}(\varrho)\,P_{m-2k,i}(t) \end{cases} \tag{8'}$$

where $\left[\frac{m-i}{2}\right]$ indicates the largest whole number contained in $\frac{m-i}{2}$.

By means of these formulas, we may calculate the partial derivatives of function Φ with respect to x, y, z. We have in fact: /79

$$\begin{cases}\dfrac{\partial\Phi}{\partial x} = \dfrac{1}{h(\varrho^2-t^2)}\left[(\varrho^2-1)\,t\,\dfrac{\partial\Phi}{\partial\varrho} + (1-t^2)\,\varrho\,\dfrac{\partial\Phi}{\partial t}\right] = \\ \qquad = \dfrac{1}{h(\varrho^2-t^2)}\displaystyle\sum_0^{\infty}{}_i\sum_{i+1}^{\infty}{}_m\,\dfrac{(m+i)\,(m-i+1)}{(2m+1)\,P_{m,i}(r)}\,(A_{m,i}\cos i\psi + B_{m,i}\,\mathrm{sen}\,i\psi)\,\times \\ \qquad\qquad \times\,[P_{m+1,i}(\varrho)\,P_{m-1,i}(t) - P_{m+1,i}(t)\,P_{m-1,i}(\varrho)], \\ \dfrac{\partial\Phi}{\partial y} = \dfrac{\sqrt{(\varrho^2-1)\,(1-t^2)}}{h(\varrho^2-t^2)}\cos\psi\left(\varrho\,\dfrac{\partial\Phi}{\partial\varrho} - t\,\dfrac{\partial\Phi}{\partial t}\right) - \dfrac{\mathrm{sen}\,\psi}{h\sqrt{(\varrho^2-1)\,(1-t^2)}}\,\dfrac{\partial\Phi}{\partial\psi} = \\ \qquad = -\,\dfrac{1}{2h(\varrho^2-t^2)}\displaystyle\sum_0^{\infty}{}_i\sum_{i+2}^{\infty}{}_m\,\dfrac{A_{m,i}\cos(i+1)\,\psi + B_{m,i}\,\mathrm{sen}\,(i+1)\,\psi}{(2m+1)\,P_{m,i}(r)\sqrt{-1}}\,\times \\ \qquad\qquad \times\,[P_{m+1,i+1}(\varrho)\,P_{m-1,i+1}(t) - P_{m+1,i+1}(t)\,P_{m-1,i+1}(\varrho)] - \\ -\,\dfrac{1}{2h(\varrho^2-t^2)}\displaystyle\sum_1^{\infty}{}_i\sum_{i}^{\infty}{}_m\,\dfrac{(m+i)\,(m+i-1)\,(m-i+1)\,(m-i+2)\,[A_{m,i}\cos(i-1)\,\psi + B_{m,i}\,\mathrm{sen}(i-1)\,\psi]}{(2m+1)\,P_{m,i}(r)\sqrt{-1}}\,\times \\ \qquad\qquad \times\,[P_{m+1,i-1}(\varrho)\,P_{m-1,i-1}(t) - P_{m+1,i-1}(t)\,P_{m-1,i-1}(\varrho)], \\ \dfrac{\partial\Phi}{\partial z} = \dfrac{\sqrt{(\varrho^2-1)\,(1-t^2)}}{h(\varrho^2-t^2)}\,\mathrm{sen}\,\psi\left(\varrho\,\dfrac{\partial\Phi}{\partial\varrho} - t\,\dfrac{\partial\Phi}{\partial t}\right) + \dfrac{\cos\psi}{h\sqrt{(\varrho^2-1)\,(1-t^2)}}\,\dfrac{\partial\Phi}{\partial\psi} = \end{cases} \tag{9}$$

$$
\left[\begin{aligned}
&= -\frac{1}{2h(\varrho^2-t^2)} \sum_0^{\infty}{}'_i \sum_{i+2}^{\infty}{}_m \frac{A_{m,i}\,\mathrm{sen}(i+1)\psi - B_{m,i}\cos(i+1)\psi}{(2m+1)\,P_{m,i}(r)\sqrt{-1}} \times \\
&\qquad \times [P_{m+1,i+1}(\varrho)\,P_{m-1,i+1}(t) - P_{m+1,i+1}(t)\,P_{m-1,i+1}(\varrho)] + \\
&+ \frac{1}{2h(\varrho^2-t^2)} \sum_1^{\infty}{}_i \sum_i^{\infty}{}_m \frac{(m+i)(m+i-1)(m-i+1)(m-i+2)[A_{m,i}\mathrm{sen}(i-1)\psi - B_{m,i}\cos(i-1)\psi]}{(2m+1)\,P_{m,i}(r)\sqrt{-1}} \times \\
&\qquad \times [P_{m+1,i-1}(\varrho)\,P_{m-1,i-1}(t) - P_{m+1,i-1}(t)\,P_{m-1,i-1}(\varrho)],
\end{aligned}\right. \tag{9}
$$

where the primes on the summations with respect to i indicate that when $i = 0$ the factor 1/2 is lacking. By means of the last equation in expression (8'), these expressions for the derivatives may be reduced to the form (7) of Φ.

2. *Solution of the problem.* Because of the statements given above, we may write

$$
\left\{\begin{aligned}
\frac{1}{4\pi}\int_\sigma u\,\frac{dG}{dn}\,d\sigma &= \sum_0^{\infty}{}_i \sum_i^{\infty}{}_m (a_{m,i}\cos i\psi + b_{m,i}\,\mathrm{sen}\,i\psi)\,\frac{P_{m,i}(\varrho)}{P_{m,i}(r)}\,P_{m,i}(t), \\
\frac{1}{4\pi}\int_\sigma v\,\frac{dG}{dn}\,d\sigma &= \sum_0^{\infty}{}_i \sum_i^{\infty}{}_m (a'_{m,i}\cos i\psi + b'_{m,i}\,\mathrm{sen}\,i\psi)\,\frac{P_{m,i}(\varrho)}{P_{m,i}(r)}\,P_{m,i}(t), \\
\frac{1}{4\pi}\int_\sigma w\,\frac{dG}{dn}\,d\sigma &= \sum_0^{\infty}{}_i \sum_i^{\infty}{}_m (a''_{m,i}\cos i\psi + b''_{m,i}\,\mathrm{sen}\,i\psi)\,\frac{P_{m,i}(\varrho)}{P_{m,i}(r)}\,P_{m,i}(t),
\end{aligned}\right. \tag{10}
$$

where a, b; a', b'; a", b" are to be considered known constants. Let us also assume that the harmonic function θ is given by the same expression (7) of Φ in such a way that the assumed values taken on by θ on σ are given by expression (5), and let us attempt to calculate the other terms which appear in expression (1). By means of equations (6) and the formulas (8) and (8'), it is easy to set the values

$$
\xi\theta = hrt\theta\,, \quad \eta\theta = h\sqrt{r^2-1}\sqrt{1-t^2}\,\theta\,, \quad \zeta\theta = h\sqrt{r^2-1}\sqrt{1-t^2}\,\theta
$$

assumed on σ in the form (5), and thus we obtain: /80

$$
\left[\begin{aligned}
\frac{1}{4\pi}\int_\sigma \xi\theta\,\frac{dG}{dn}\,d\sigma &= hr\sum_0^{\infty}{}_i \sum_i^{\infty}{}_m \Bigg[\left(\frac{m+i+1}{2m+3}A_{m+1,i} + \frac{m-i}{2m-1}A_{m-1,i}\right)\cos i\psi + \\
&\quad + \left(\frac{m+i+1}{2m+3}B_{m+1,i} + \frac{m-i}{2m-1}B_{m-1,i}\right)\mathrm{sen}\,i\psi\Bigg]\frac{P_{m,i}(\varrho)}{P_{m,i}(r)}\,P_{m,i}(t), \\
\frac{1}{4\pi}\int_\sigma \eta\theta\,\frac{dG}{dn}\,d\sigma &= \frac{h}{2}\sqrt{r^2-1}\sum_0^{\infty}{}_i \sum_i^{\infty}{}_m \Bigg\{\Bigg[\frac{(m+i+1)(m+i+2)}{2m+3}A_{m+1,i+1} - \\
&\quad - \frac{(m-i-1)(m-i)}{2m-1}A_{m-1,i+1} - \frac{A_{m+1,i-1}}{2m+3} + \frac{A_{m-1,i-1}}{2m-1}\Bigg]\cos i\psi + \\
&\quad + \Bigg[\frac{(m+i+1)(m+i+2)}{2m+3}B_{m+1,i+1} - \frac{(m-i-1)(m-1)}{2m-1}B_{m-1,i+1} - \\
&\quad - \frac{B_{m+1,i-1}}{2m+3} + \frac{B_{m-1,i-1}}{2m-1}\Bigg]\mathrm{sen}\,i\psi\Bigg\}\frac{P_{m,i}(\varrho)}{P_{m,i}(r)}\,P_{m,i}(t), \\
\frac{1}{4\pi}\int_\sigma \zeta\theta\,\frac{dG}{dn}\,d\sigma &= \frac{h}{2}\sqrt{r^2-1}\sum_0^{\infty}{}_i \sum_i^{\infty}{}_m \Bigg\{\Bigg[\frac{(m+i+1)(m+i+2)}{2m+3}B_{m+1,i+1} -
\end{aligned}\right. \tag{11}
$$

$$
\left.\begin{aligned}
&-\frac{(m-i-1)(m-i)}{2m-1}\,B_{m-1,i+1}+\frac{B_{m+1,i-1}}{2m+3}-\frac{B_{m-1,i-1}}{2m-1}\Big]\cos i\psi-\\
&-\Big[\frac{(m+i+1)(m+i+2)}{2m+3}\,A_{m+1,i+1}-\frac{(m-i-1)(m-i)}{2m-1}\,A_{m-1,i+1}+\\
&\qquad+\frac{A_{m+1,i-1}}{2m+3}-\frac{A_{m-1,i-1}}{2m-1}\Big]\operatorname{sen} i\psi\Big\}\frac{P_{m,i}(\varrho)}{P_{m,i}(r)}\,P_{m,i}(t),
\end{aligned}\right. \tag{11}
$$

so that we may write

$$
\left\{\begin{aligned}
u=&\sum_0^\infty{}_i\sum_i^\infty{}_m\Big\{\Big[a_{m,i}-\frac{\lambda+\mu}{2\mu}\Big(xA_{m,i}-hr\Big(\frac{m+i+1}{2m+3}\,A_{m+1,i}+\frac{m-i}{2m-1}\,A_{m-1,i}\Big)\Big)\Big]\cos i\psi+\\
&+\Big[b_{m,i}-\frac{\lambda+\mu}{2\mu}\Big(xB_{m,i}-hr\Big(\frac{m+i+1}{2m+3}\,B_{m+1,i}+\frac{m-i}{2m-1}\,B_{m-1,i}\Big)\Big)\Big]\operatorname{sen} i\psi\Big\}\frac{P_{m,i}(\varrho)}{P_{m,i}(r)}P_{m,i}(t),\\
v=&\sum_0^\infty{}_i\sum_i^\infty{}_m\Big\{\Big[a'_{m,i}-\frac{\lambda+\mu}{2\mu}\Big(yA_{m,i}-\frac{h}{2}\sqrt{r^2-1}\Big(\frac{(m+i+1)(m+i+2)}{2m+3}\,A_{m+1,i+1}-\\
&-\frac{(m-i-1)(m-i)}{2m-1}\,A_{m-1,i+1}-\frac{A_{m+1,i-1}}{2m+3}+\frac{A_{m-1,i-1}}{2m-1}\Big)\Big)\Big]\cos i\psi+\\
&+\Big[b'_{m,i}-\frac{\lambda+\mu}{2\mu}\Big(yB_{m,i}-\frac{h}{2}\sqrt{r^2-1}\Big(\frac{(m+i+1)(m+i+2)}{2m+3}\,B_{m+1,i+1}-\\
&-\frac{(m-i-1)(m-i)}{2m-1}\,B_{m-1,i+1}-\frac{B_{m+1,i-1}}{2m+3}+\frac{B_{m-1,i-1}}{2m-1}\Big)\Big)\Big]\operatorname{sen} i\psi\Big\}\frac{P_{m,i}(\varrho)}{P_{m,i}(r)}\,P_{m,i}(t),\\
w=&\sum_0^\infty{}_i\sum_i^\infty{}_m\Big\{\Big[a''_{m,i}-\frac{\lambda+\mu}{2\mu}\Big(zA_{m,i}-\frac{h}{2}\sqrt{r^2-1}\Big(\frac{(m+i+1)(m+i+2)}{2m+3}\,B_{m+1,i+1}-\\
&-\frac{(m-i-1)(m-i)}{2m-1}\,B_{m-1,i+1}+\frac{B_{m+1,i-1}}{2m+3}-\frac{B_{m-1,i-1}}{2m-1}\Big)\Big)\Big]\cos i\psi+\\
&+\Big[b''_{m,i}-\frac{\lambda+\mu}{2\mu}\Big(zB_{m,i}+\frac{h}{2}\sqrt{r^2-1}\Big(\frac{(m+i+1)(m+i+2)}{2m+3}\,A_{m+1,i+1}-\\
&-\frac{(m-i-1)(m-i)}{2m-1}\,A_{m-1,i+1}+\frac{A_{m+1,i-1}}{2m+3}-\frac{A_{m-1,i-1}}{2m-1}\Big)\Big)\Big]\operatorname{sen} i\psi\Big\}\frac{P_{m,i}(\varrho)}{P_{m,i}(r)}\,P_{m,i}(t).
\end{aligned}\right. \tag{12}
$$

The problem is now reduced to determining the constants A, B by means of the known constants a, b; a', b'; a", b", so that equation (2) is identically satisfied. When this equation is now multiplied by ρ^2-t^2 and it is seen that /81

$$
\begin{aligned}
(\varrho^2-t^2)\,\theta=&\sum_0^\infty{}_i\sum_i^\infty{}_m\frac{(m-i+1)(m-i+2)}{(2m+1)(2m+3)\,P_{m,i}(r)}\,(A_{m,i}\cos i\psi+B_{m,i}\operatorname{sen} i\psi)\times\\
&\times[P_{m+2,i}(\varrho)\,P_{m,i}(t)-P_{m+2,i}(t)\,P_{m,i}(\varrho)]\\
&-\sum_0^\infty{}_i\sum_{i+2}^\infty{}_m\frac{(m+i)(m+i-1)}{(2m-1)(2m+1)\,P_{m,i}(r)}\,(A_{m,i}\cos i\psi+B_{m,i}\operatorname{sen} i\psi)\times\\
&\times[P_{m,i}(\varrho)\,P_{m-2,i}(t)-P_{m,i}(t)\,P_{m-2,i}(\varrho)],\\
(\varrho^2-t^2)\Big(x\frac{\partial\theta}{\partial x}+y\frac{\partial\theta}{\partial y}+z\frac{\partial\theta}{\partial z}\Big)=&(\varrho^2-1)\,\varrho\frac{\partial\theta}{\partial\varrho}+(1-t^2)\,t\frac{\partial\theta}{\partial t}=\\
=&\sum_0^\infty{}_i\sum_i^\infty{}_m\frac{m(m-i+1)(m-i+2)}{(2m+1)(2m+3)\,P_{m,i}(r)}\,(A_{m,i}\cos i\psi+B_{m,i}\operatorname{sen} i\psi)\times\\
&\times[P_{m+2,i}(\varrho)\,P_{m,i}(t)-P_{m+2,i}(t)\,P_{m,i}(\varrho)]+
\end{aligned}
$$

$$+\sum_{0}^{\infty}{}_{i}\sum_{i+2}{}_{m}\frac{(m+1)(m+i)(m+i-1)}{(2m-1)(2m+1)P_{m,i}(r)}(A_{m,i}\cos i\psi+B_{m,i}\operatorname{sen} i\psi)\times$$
$$\times[P_{m,i}(\varrho)P_{m-2,i}(t)-P_{m,i}(t)P_{m-2,i}(\varrho)],$$

it is immediately found that whenever equation (2) is involved we may write:

$$\left.\begin{aligned}&\sum_{0}^{\infty}{}'_{i}\sum_{i}^{\infty}{}_{m}\frac{1}{2m+3}\left\{\left[\frac{(m-i+1)(m-i+2)}{2m+1}R_{m,i}A_{m,i}+\right.\right.\\&\quad\left.+\frac{(m+i+1)(m+i+2)}{2m+5}S_{m+2,i}A_{m+2,i}-T_{m,i}\right]\cos i\psi+\\&\quad+\left[\frac{(m-i+1)(m-i+2)}{2m+1}R_{m,i}B_{m,i}+\right.\\&\quad\left.\left.+\frac{(m+i+1)(m+i+2)}{2m+5}S_{m+2,i}B_{m+2,i}-T'_{m,i}\right]\operatorname{sen} i\psi\right\}\times\\&\quad\times[P_{m+2,i}(\varrho)P_{m,i}(t)-P_{m+2,i}(t)P_{m,i}(\varrho)]=0\end{aligned}\right\}\qquad(13)$$

in which

$$\left.\begin{aligned}R_{m,i}&=\frac{3\lambda+5\mu+m(\lambda+\mu)}{(\lambda+\mu)P_{m,i}(r)}-r\frac{m+i+1}{P_{m+1,i}(r)}+\\&\quad+\frac{\sqrt{1-r^2}}{2P_{m+1,i-1}(r)}-\frac{(m+i+1)(m+i+2)\sqrt{1-r^2}}{2P_{m+1,i+1}(r)},\\S_{m+2,i}&=\frac{-3\lambda-5\mu+(m+3)(\lambda+\mu)}{(\lambda+\mu)P_{m+2,i}(r)}-r\frac{m-i+2}{P_{m+1,i}(r)}-\\&\quad-\frac{\sqrt{1-r^2}}{2P_{m+1,i-1}(r)}+\frac{(m-i+1)(m-i+2)\sqrt{1-r^2}}{2P_{m+1,i+1}(r)},\\T_{m,i}&=\frac{\mu}{h(\lambda+\mu)}\left[2(m+i+1)(m-i+2)\frac{a_{m+1,i}}{P_{m+1,i}(r)}-\frac{a'_{m+1,i-1}-b''_{m+1,i-1}}{\sqrt{-1}\,P_{m+1,i-1}(r)}-\right.\\&\quad\left.-\frac{(m+i+1)(m+i+2)(m-i+1)(m-i+2)(a'_{m+1,i+1}+b''_{m+1,i+1})}{\sqrt{-1}\,P_{m+1,i+1}(r)}\right],\\T'_{m,i}&=\frac{\mu}{h(\lambda+\mu)}\left[2(m+i+1)(m-i+2)\frac{b_{m+1,i}}{P_{m+1,i}(r)}-\frac{b'_{m+1,i-1}+a''_{m+1,i-1}}{\sqrt{-1}\,P_{m+1,i-1}(r)}-\right.\\&\quad\left.-\frac{(m+i+1)(m+i+2)(m-i+1)(m-i+2)(b'_{m+1,i+1}-a''_{m+1,i+1})}{\sqrt{-1}\,P_{m+1,i+1}(r)}\right],\end{aligned}\right\}\qquad(14)$$

and the prime on the summation with respect to i indicates that, for the first /82
two values of i: ($R_{m,o}$, $R_{m,1}$), the values of $S_{m+2,0},\ldots$ depart somewhat from the general law.

If we now proceed to divide by ρ^2-t^2 and take into account the last equation in expression (8'), then formula (13) will easily assume the form

$$\sum_{0}^{\infty}{}'_{i}\sum_{i}^{\infty}{}_{m}\left\{\left[R_{m,i}A_{m,i}+(2m+1)\frac{(m-i)!}{(m+i)!}\sum_{1}^{\infty}{}_{k}\frac{(m+i+2k)!}{(m-i+2k)!}\times\right.\right.$$
$$\times(S_{m+2k,i}+R_{m+2k,i})\frac{A_{m+2k,i}}{2m+4k+1}-\qquad(13')$$
$$\left.-(2m+1)\frac{(m-i)!}{(m+i)!}\sum_{0}^{\infty}{}_{k}\frac{(m+i+2k)!}{(m-i+2k+2)!}T_{m+2k,i}\right]\cos i\psi+$$

$$+\Big[R_{m,i}\,B_{m,i}+(2m+1)\frac{(m-i)!}{(m+i)!}\sum_1^\infty{}_k\frac{(m+i+2k)!}{(m-i+2k)!}\times$$

$$\times(S_{m+2k,i}+R_{m+2k,i})\frac{B_{m+2k,i}}{2m+4k+1}- \tag{13'}$$

$$-(2m+1)\frac{(m-i)!}{(m+i)!}\sum_0^\infty{}_k\frac{(m+i+2k)!}{(m-i+2k+2)!}T'_{m+2k,i}\Big]\operatorname{sen} i\psi\Big\}P_{m,i}(\varrho)\,P_{m,i}(t)=0.$$

To determine the values of A corresponding to a given value of subscript i, we hence have the equations

$$\left\{\begin{aligned}&\frac{(m+i)!}{(m-i)!}R_{m,i}\frac{A_{m,i}}{2m+1}+\sum_1^\infty{}_k\frac{(m+i+2k)!}{(m-i+2k)!}\times\\&\qquad\times(S_{m+2k,i}+R_{m+2k,i})\frac{A_{m+2k,i}}{2m+4k+1}=\\&=\sum_0^\infty{}_k\frac{(m+i+2k)!}{(m-i+2k+2)!}T_{m+2k,i}\,,\quad m=i\,,\,i+1\cdots\infty\end{aligned}\right. \tag{15}$$

and we have similar equations to find the values of B corresponding to that value of subscript i. We thus arrive at the theory of infinite determinants. We may, however, avoid using this theory by noting that--if the coefficients of cos $i\psi$ and sin $i\psi$ in expression (13) are zero--the terms of expression (15) are identically satisfied. Since we know that the problem has but a single solution, we may conclude that our unknowns include those which satisfy the following equations:

$$\begin{aligned}&\frac{(m-i+1)(m-i+2)}{2m+1}R_{m,i}A_{m,i}+\\&\qquad+\frac{(m+i+1)(m+i+2)}{2m+5}S_{m+2,i}A_{m+2,i}=T_{m,i}\end{aligned} \tag{16}$$

and similar equations in terms of B. Then from expression (16) we easily derive:

$$\left\{\begin{aligned}&\frac{(m+i+2k)!}{(m-i+2k)!}\frac{A_{m+2k,i}}{2m+4k+1}=\\&=(-1)^k\frac{(m+i)!}{(m-i)!}\frac{R_{m+2k-2,i}}{S_{m+2k,i}}\cdot\frac{R_{m+2k-4,i}}{S_{m+2k-2,i}}\cdots\frac{R_{m,i}}{S_{m+2,i}}\frac{A_{m,i}}{2m+1}+\\&\qquad+\sum_1^k{}_h(-1)^{h+1}\frac{(m+2k-2h+i)!}{(m+2k-2h+2-i)!}\cdot\\&\qquad\cdot\frac{R_{m+2k-2,i}}{S_{m+2k,i}}\cdot\frac{R_{m+2k-4,i}}{S_{m+2k-2,i}}\cdots\frac{R_{m+2k-2h+2,i}}{S_{m+2k-2h+4,i}}\cdot\frac{T_{m+2k-2h,i}}{S_{m+2k-2h+2,i}}\,,\end{aligned}\right. \tag{17}$$

Thus, by substitution into expression (15) we directly have /83

$$\begin{aligned}&\Big[R_{m,i}+\sum_1^\infty{}_k(-1)^k(S_{m+2k,i}+R_{m+2k,i})\frac{R_{m+2k-2,i}}{S_{m+2k,i}}\cdot\\&\qquad\cdot\frac{R_{m+2k-4,i}}{S_{m+2k-2,i}}\cdots\frac{R_{m,i}}{S_{m+2,i}}\Big]\frac{(m+i)!}{(m-i)!}\frac{A_{m,i}}{2m+1}=\\&=\sum_0^\infty{}_k\frac{(m+i+2k)!}{(m-i+2k+2)!}T_{m+2k,i}-\sum_1^\infty{}_k\sum_1^k{}_h(-1)^{h+1}\frac{(m+2k-2h+i)!}{(m+2k-2h+2-i)!}\times\end{aligned} \tag{18}$$

$$\times (S_{m+2k,i} + R_{m+2k,i}) \frac{R_{m+2k-2,i}}{S_{m+2k,i}} \cdot \frac{R_{m+2k-4,i}}{S_{m+2k-2,i}} \cdots \frac{R_{m+2k-2h+2,i}}{S_{m+2k-2h+4,i}} \cdot \frac{T_{m+2k-2h,i}}{S_{m+2k-2h+2,i}} . \tag{18}$$

From similar formulas we may determine the values of $B_{m,i}$. At an early opportunity, we propose to examine the validity of this solution in greater detail.

3. *Regarding the case in which the surface stresses are given.* The new problem, if we wish to handle it directly with the principles which I have employed in other cases, will present difficulties having no easy solution. The simplest method to solve it is, in my opinion, to formulate the expressions:

$$\left\{\begin{aligned} &\lambda\theta \frac{x}{r^2} + 2\mu\left[\frac{\partial u}{\partial x}\frac{x}{r^2} + \frac{1}{r^2-1}\left(y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + y\varpi_3 - z\varpi_2\right)\right], \\ &\lambda\theta \frac{y}{r^2-1} + 2\mu\left[\frac{x}{r^2}\left(\frac{\partial u}{\partial x} - \varpi_3\right) + \frac{1}{r^2-1}\left(y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + z\varpi_1\right)\right], \\ &\lambda\theta \frac{z}{r^2-1} + 2\mu\left[\frac{x}{r^2}\left(\frac{\partial u}{\partial x} + \varpi_2\right) + \frac{1}{r^2-1}\left(y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} - y\varpi_1\right)\right], \end{aligned}\right. \tag{19}$$

where the values of u, v, w are those found in the preceding problem and to transform these expressions suitably so that their surface values are developed into a series of spherical functions. This may be done by utilizing expressions (8) and (8'). Seeing that the values thus obtained are also the values of

$$L\sqrt{\frac{x^2}{r^4} + \frac{y^2+z^2}{(r^2-1)^2}}\ ,\quad M\sqrt{\frac{x^2}{r^4} + \frac{y^2+z^2}{(r^2-1)^2}}\ ,\quad N\sqrt{\frac{x^2}{r^4} + \frac{y^2+z^2}{(r^2-1)^2}} \tag{20}$$

where L, M, N are the given surface stresses, we then have the task of expanding expressions (20) into a series of spherical functions and of determining the constants a, b; a',b'; a", b" which appear in the expressions of u, v, w, so that on the surface of the ellipsoid expressions (19) become equal to the corresponding expressions (20). /84

4. *Remarks.* These procedures, which as a particular case contain a new method for solving elastic equilibrium problems for the sphere and for two concentric spheres, may obviously be extended to solve the problem of elastic equilibrium for a body limited by two confocal ellipsoids of rotation.

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